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DARBOUX INTEGRABILITY OF 2–DIMENSIONAL HAMILTONIAN SYSTEMS WITH HOMOGENOUS POTENTIALS OF DEGREE 3

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Abstract. We provide a characterization of all Hamiltonian systems of the form $H = (p_1^2 + p_2^2)/2 + V(q_1, q_2)$ where V is a homogenous polynomial of degree 3 which are completely integrable with Darboux first integrals.

1. Introduction and statement of the main results

We consider $\mathbb{R}^4$ as a symplectic linear space with canonical variables $q = (q_1, q_2)$ and $p = (p_1, p_2)$, with q_i called the *coordinates* and p_i called the *momenta*. We want to study the Hamiltonian systems with Hamilton's function of the form

$$H = \frac{1}{2}\sum_{i=1}^{2} p_i^2 + V(q),$$

where $V(q) = V(q_1, q_2)$ is a homogeneous polynomial of degree 3, i.e. we will study the Hamiltonian systems

$$\dot{q}_i = p_i, \quad \dot{p}_i = -\frac{\partial V}{\partial q_i}, \quad i = 1, 2. \tag{1}$$

During the last century these systems have been actively investigated, with special attention to these systems where V has degree at most 5 and which have a second polynomial first integral with degree at most 4 in the variables p_1 and p_2 (see for instance [1, 2, 10, 11, 18]). When the homogeneous potential has degree three we want to mention the work [12] where he studied the polynomial integrability assuming that the additional first integral is a polynomial of order at most four in the momenta.

We start by recalling some definitions. Let $A = A(p, q)$ and $B = B(p, q)$ be two functions. Their *Poisson bracket* $\{A, B\}$ is defined as

$$\{A, B\} = \sum_{i=1}^{2}\left(\frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i}\right).$$

We say that two functions A and B are *in involution* if $\{A, B\} = 0$. We say that a non-constant function $F = F(q, p)$ is a *first integral* for the Hamiltonian system (1) if it commutes with the Hamiltonian function H, i.e., $\{H, F\} = 0$. We say that the Hamiltonian system (1) is *completely integrable* if it has 2 functionally

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independent first integrals which are in involution. One first integral will always be the Hamiltonian H. We say that two functions H and F are *independent* if their gradients are linearly independent at all points $\mathbb{R}^4$ except perhaps in a zero Lebesgue set.

A *polynomial first integral* $f = f(x, y, p_x, p_y)$ of system (1) is a first integral which is a polynomial, and a *rational first integral* of system (1) is a first integral which is a rational function. To study the existence of rational first integrals we will use the well-known Darboux theory of integrability. The Darboux theory of integrability in dimension 4 is based on the existence of invariant algebraic hypersurfaces (or Darboux polynomials). For more details see [4, 5] and [13]. This theory is one of the best theories for studying the existence of first integrals for the polynomial differential systems.

A *Darboux polynomial* of system (1) is a polynomial $f \in \mathbb{C}[p_1, p_2, q_1, q_2] \setminus \mathbb{C}$ such that

$$\sum_{i=1}^{2} \left(p_i \frac{\partial f}{\partial q_i} - \frac{\partial V}{\partial q_i} \frac{\partial f}{\partial p_i} \right) = Kf \tag{2}$$

for some polynomial K called the *cofactor* of f and with degree at most one.

Note that $f = 0$ is an invariant algebraic hypersurface for the flow of system (1), and a *polynomial first integral* if it is a Darboux polynomial with zero cofactor. We recall that if $f \notin \mathbb{R}[p_1, p_2, q_1, q_2] \setminus \mathbb{R}$ is a Darboux polynomial then there exists another Darboux polynomial $\bar{f}$ (the conjugate of f) with cofactor $\bar{K}$ (the conjugate of K).

An *exponential factor* $F = F(p_1, p_2, q_1, q_2)$ of system (1) is a function of the form $F = \exp(g_0/g_1) \notin \mathbb{C}$ with $g_0, g_1 \in \mathbb{C}[p_1, p_2, q_1, q_2]$ coprime satisfying that

$$\sum_{i=1}^{2} \left(p_i \frac{\partial F}{\partial q_i} - \frac{\partial V}{\partial q_i} \frac{\partial F}{\partial p_i} \right) = LF,$$

for some polynomial $L = L(p_1, p_2, q_1, q_2)$ called the *cofactor* of F and with degree at most one. We recall that if $F \notin \mathbb{R}[p_1, p_2, q_1, q_2] \setminus \mathbb{R}$ is an exponential factor then there exists another exponential factor $\bar{F}$ (the conjugate of F) with cofactor $\bar{L}$ (the conjugate of L).

A *Darboux first integral* G of system (1) is a first integral of the form

$$G = f_1^{\lambda_1} \cdots f_p^{\lambda_p} F_1^{\mu_1} \cdots F_q^{\mu_q}, \tag{3}$$

where $f_1, \ldots, f_p$ are Darboux polynomials and $F_1, \ldots, F_q$ are exponential factors and $\lambda_j, \mu_k \in \mathbb{C}$ for $j = 1, \ldots, p$ and $k = 1, \ldots, q$. Note that a Darboux first integral is always a real function due to the fact that if there are complex Darboux polynomials or complex exponential factors always also appear their conjugates.

We consider the Hamiltonian systems (1) with a homogeneous potential of degree 3 of the form

$$V = V(q_1, q_2) = \frac{a}{3} q_1^2 + \frac{b}{2} q_1^2 q_2 + c q_1 q_2^2 + \frac{d}{3} q_2^3, \quad a, b, c, d \in \mathbb{R} \tag{4}$$

Let $\mathrm{PO}_2(\mathbb{C})$ denote the group of 2×2 complex matrices A such that $AA^T = \alpha\,\mathrm{Id}$, where Id is the identity matrix and $\alpha \in \mathbb{C} \setminus \{0\}$. The potentials $V_1(\mathbf{q})$ and $V_2(\mathbf{q})$ are *equivalent* if there exists a matrix $A \in \mathrm{PO}_2(\mathbb{C})$ such that $V_1(\mathbf{q}) = V_2(A\mathbf{q})$.

Therefore we divide all potentials into equivalent classes. In what follows a potential means a class of equivalent potentials in the above sense. This definition of equivalent potentials is motivated by the following simple observation (for a proof see [12]). Let V_1 and V_2 be two equivalent potentials, if the Hamiltonian system with Hamiltonian

$$H = \frac{1}{2}\sum_{i=1}^{2} p_i^2 + V(q_1, q_2), \tag{5}$$

and potential $V = V_1$ is integrable, then it is also integrable with the potential $V = V_2$.

A very relevant result related with the integrability of Hamiltonian systems with homogeneous potential was given by Morales and Ramis (see page 100 of [17] and the references therein), with a method that gives a necessary condition for the existence of an additional meromorphic first integral. Using these results, Maciejewski and Przybylska [15, 16] proved the following result.

Theorem 1. *The Hamiltonian systems* (1) *with homogeneous potential of degree* 3 *is completely integrable with meromorphic first integrals if and only if* V *is one of the potentials*

$$\begin{aligned}
V_k &= a(q_2 - iq_1)^k (q_2 + iq_1)^{3-k} \quad \textit{for } k = 0, \dots, 3,\\
V_4 &= q_1^3,\\
V_5 &= q_1^3/3 + bq_2^3/3,\\
V_6 &= q_1^2 q_2/2 + q_2^3,\\
V_7 &= q_1^2 q_2/2 + 8q_2^3/3,\\
V_8^{\pm} &= \pm i\sqrt{3}q_1^3/18 + q_1^2 q_2/2 + q_2^3.
\end{aligned} \tag{6}$$

Unfortunately Theorem 1 is not completely correct because the potential

$$V_9 = \frac{a}{3}q_1^3 + \frac{1}{2}q_1^2 q_2 + \frac{1}{6}q_2^3 \quad \text{with} \quad a \in \mathbb{R} \tag{7}$$

has an additional polynomial first integral given in Table 1, which of course is a meromorphic first integral. We find this new integrable case in the proof of Proposition 7 studying the exponential factors of system (1).

Our main result is the following one.

Theorem 2. *The Hamiltonian systems* (1) *with homogeneous potential of degree* 3 *is completely integrable with Darboux first integrals if and only if* V *is one of the potentials given in Table 1 with* $V_0, V_1, \dots, V_8^{\pm}$ *given in* (6) *and* V_9 *given in* (7). *In Table 1 we also provide the additional independent polynomial first integral for each potential.*

We recall that not all meromorphic functions are Darboux functions, and that not all Darboux functions are meromorphic functions. This shows that Theorems 1 and 2 are independent.

In section 2 we provide a brief introduction to the quasi-homogenous polynomial differential systems that will allow to prove Theorem 2. In section 3 we give the proof of Theorem 2. We recall that since we cannot find in the literature the explicit expressions of all these polynomial first integrals for the nine potentials of Table

Potential	First integral
V_0	$H_0 = p_1 - p_2 i$,
V_1	$H_1 = 9ip_1^2 + 6p_1p_2 + 3ip_2^2 - 16aq_1^3 + 24iaq_1^2q_2 + 8iaq_2^3$,
V_2	$H_2 = -9ip_1^2 + 6p_1p_2 - 3ip_2^2 - 16aq_1^3 - 24iaq_1^2q_2 - 8iaq_2^3$,
V_3	$H_3 = p_1 + ip_2$,
V_4	$H_4 = p_2$,
V_5	$H_5 = 3p_1^2 + 2q_1^3$,
V_6	$H_6 = -8p_1p_2q_1 + 8p_1^2q_2 - q_1^2(q_1^2 + 4q_2^2)$,
V_7	$H_7 = 72p_1^4 - 36p_1p_2q_1^3 - 3q_1^6 + 2(3p_2^2 + 16q_2^3)(3p_2^2 + 6q_1^2q_2 + 16q_2^3)$
	$+12p_1^2(3p_2^2 + 12q_1^2q_2 + 16q_2^3)$,
$V_8^{\pm}$	$H_8^{\pm} = -q_1^6 \pm 6\sqrt{3}iq_1^5q_2 + 27q_1^4q_2^2 \pm 6\sqrt{3}iq_1^3(p_1^2 + p_2^2 + 2q_2^3)$
	$+54q_1^2q_2(p_1^2 + p_2^2 + 2q_2^3) + 27(p_1^2 + p_2^2 + 2q_2^3)^2$,
V_9	$H_9 = ap_1^2 + p_1p_2 + q_1^3/6 + 2a^2q_1^3/3 + aq_1^2q_2 + q_1q_2^2/2$.

Table 1. All nonequivalent integrable homogeneous polynomial potentials of degree 3 with their corresponding additional first integral. Here $i = \sqrt{-1}$.

1, we shall provide them here, we have used for their computation the theory of integrability for the quasi–homogeneous polynomial differential systems.

2. Quasi–homogeneous polynomial differential systems

In this subsection we summarize some basic results on the analytic and polynomial integrability of the polynomial differential systems of the form

$$\frac{d\mathbf{x}}{dt} = \dot{\mathbf{x}} = \mathbf{P}(\mathbf{x}), \qquad \mathbf{x} = (x_1, \cdots, x_n) \in \mathbb{C}^n, \tag{8}$$

with $\mathbf{P}(\mathbf{x}) = (P_1(\mathbf{x}), \cdots, P_n(\mathbf{x}))$ and $P_i \in \mathbb{C}[x_1, \cdots, x_n]$ for $i = 1, \cdots, n$. As usual $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ will denote the sets of positive integers, real and complex numbers, respectively; and $\mathbb{C}[x_1, \cdots, x_n]$ denotes the polynomial ring over $\mathbb{C}$ in the variables $x_1, \cdots, x_n$. Here t can be real or complex.

The polynomial differential system (8) is *quasi–homogeneous* if there exist $\mathbf{s} = (s_1, \cdots, s_n) \in \mathbb{N}^n$ and $d \in \mathbb{N}$ such that for arbitrary $\alpha \in \mathbb{R}^+ = \{a \in \mathbb{R}, a > 0\}$,

$$P_i(\alpha^{s_1}x_1, \cdots, \alpha^{s_n}x_n) = \alpha^{s_i - 1 + d}P_i(x_1, \cdots, x_n),$$

for $i = 1, \ldots, n$. We call $\mathbf{s} = (s_1, \cdots, s_n)$ the *weight exponent* of system (8), and d the *weight degree* with respect to the weight exponent $\mathbf{s}$. In the particular case that $\mathbf{s} = (1, \cdots, 1)$ system (8) is the classical *homogeneous polynomial differential system of degree d*.

Note that if the polynomial differential system (8) is quasi–homogeneous with weight exponent $\mathbf{s}$ and weight degree $d > 1$, then the system is invariant under the change of variables $x_i \to \alpha^{w_i}x_i$, $t \to \alpha^{-1}t$, where $w_i = s_i/(d-1)$.

Recently the integrability of quasi–homogeneous polynomial differential systems have been investigated by several authors. Probably the best results have been provided by Yoshida [20, 21, 22], Furta [7] and Goriely [9], see also Tsygvintsev [19] and Llibre and Zhang [13].

A non–constant function $F(x_1,\ldots,x_n)$ is a *first integral* of system (8) if it is constant on all solution curves $(x_1(t),\cdots,x_n(t))$ of system (8); i.e. $F(x_1(t),\cdots,x_n(t)) =$ constant for all values of t for which the solution $(x_1(t),\cdots,x_n(t))$ is defined. If F is C^1, then F is a first integral of system (8) if and only if

$$\sum_{i=1}^{n} P_i \frac{\partial F}{\partial x_i} \equiv 0.$$

The function $F(x_1,\ldots,x_n)$ is *quasi–homogeneous of weight degree* m *with respect to the weight exponent* $\mathbf{s}$ if it satisfies

$$F(\alpha^{s_1}x_1,\ldots,\alpha^{s_n}x_n) = \alpha^m F(x_1,\ldots,x_n),$$

for all $\alpha\in\mathbb{R}^+$.

Given an analytic function F we can split it in the form $F=\sum_i F^i$, where F^i is a quasi–homogeneous polynomial of weight degree i with respect to the weight exponent $\mathbf{s}$; i.e. $F^i(\alpha^{s_1}x_1,\ldots,\alpha^{s_n}x_n) = \alpha^i F^i(x_1,\ldots,x_n)$. The following result is well known, see for instance Proposition 1 of [13].

Proposition 3. *Let F be an analytic function and let $F=\sum_i F^i$ be its decomposition into weight–homogeneous polynomials of weight degree i with respect to the weight exponent* $\mathbf{s}$. *Then F is an analytic first integral of the weight–homogeneous polynomial differential system* (8) *with weight exponent* $\mathbf{s}$ *if and only if each weight homogeneous part F^i is a first integral of system* (8) *for all i.*

Suppose that system (8) is a quasi–homogeneous polynomial differential system of weight degree d with respect to the weight exponent $\mathbf{s}$. Then we define $\mathbf{w} = \mathbf{s}/(d-1)$. The interest for the quasi–homogeneous polynomial differential systems is based in the existence of the particular solutions of the form

$$(x_1(t),\ldots,x_n(t)) = \left(c_1t^{-w_1},\ldots,c_nt^{-w_n}\right),$$

where the coefficients $\mathbf{c}=(c_1,\ldots,c_n)\in\mathbb{C}^n\setminus\{\mathbf{0}\}$ are given by the algebraic system of equations

$$P_i(c_1,\ldots,c_n)+w_ic_i=0 \quad \text{for} \quad i=1,\ldots,n.$$

For a given $(w_1,\ldots,w_n)$ there may exist different $\mathbf{c}$'s, called the *balances*.

For each balance $\mathbf{c}$ we introduce a matrix

$$K(\mathbf{c}) = D\mathbf{P}(\mathbf{c}) + \operatorname{diag}(w_1,\ldots,w_n),$$

where as usual $D\mathbf{P}(\mathbf{c})$ denotes the differential of $\mathbf{P}$ evaluated at $\mathbf{c}$, and $\operatorname{diag}(w_1,\ldots,w_n)$ denotes the matrix whose diagonal is equal to $(w_1,\ldots,w_n)$ and zeros in the rest.

The eigenvalues of $K(\mathbf{c})$ are called the *Kowalevsky exponents* of the balance $\mathbf{c}$. Sophia Kowalevskaya was the first to introduce the matrix K to compute the Laurent series solutions of the rigid body motion. It can be shown that there always exists a Kowalevsky exponent equal to -1 related to the arbitrariness of the origin of the parametrization of the solution by the time. The eigenvector associated to the eigenvalue -1 is $\mathbf{wc}=(w_1c_1,\,\ldots,\,w_nc_n)$, for more details see [20] or [7].

A non–constant function $F(x_1,\ldots,x_n)$ is a *first integral* of system (8) if it is constant on all solution curves $(x_1(t),\cdots,x_n(t))$ of system (8); i.e. $F(x_1(t),\cdots,x_n(t)) =$

constant for all values of t for which the solution $(x_1(t), \cdots, x_n(t))$ is defined. If F is C^1, then F is a first integral of system (8) if and only if

$$\sum_{i=1}^{n} P_i \frac{\partial F}{\partial x_i} \equiv 0.$$

Probably the best results in order to know if a weighted homogeneous polynomial of weight degree m with respect to the weight exponent $\mathbf{s}$ is a first integral of a quasi–homogeneous polynomial differential system (8) with weight degree d with respect to the weight exponent $\mathbf{s}$ is essentially due to Yoshida [20], and are the following two theorems.

Theorem 4. *Let $F(x_1, \ldots, x_n)$ be a weighted homogeneous polynomial first integral of weight degree m with respect to the weight exponent $\mathbf{s}$ of the quasi–homogeneous polynomial differential system (8) with weight degree d with respect to the weight exponent $\mathbf{s}$. Suppose the gradient of F evaluated at a balance $\mathbf{c}$ is finite and not identically zero. Then $m/(d-1)$ is a Kowalevsky exponent of the balance $\mathbf{c}$.*

Let r be a positive integer such that $1 < r < n$. A set of functions $F_k : \mathbb{C}^n \to \mathbb{C}$ for $k = 1, \ldots, r$ are *independent* if the rank of the $r \times n$ matrix

$$\begin{pmatrix} \dfrac{\partial F_1}{\partial x_1} & \cdots & \dfrac{\partial F_1}{\partial x_n} \\ \cdots & \cdots & \cdots \\ \dfrac{\partial F_r}{\partial x_1} & \cdots & \dfrac{\partial F_r}{\partial x_n} \end{pmatrix} (\mathbf{x})$$

is r for all the $\mathbf{x} \in \mathbb{C}^n$, except perhaps in a zero Lebesgue subset of $\mathbb{C}^n$.

Theorem 5. *Let r be a positive integer such that $1 < r < n$, and let $F_k(x_1, \ldots, x_n)$ for $k = 1, \ldots, r$ be weighted homogeneous polynomial first integrals of weight degree m with respect to the weight exponent $\mathbf{s}$ of the quasi–homogeneous polynomial differential system (8) with weight degree d with respect to the weight exponent $\mathbf{s}$.*

Suppose that the gradients of F_k for $k = 1, \ldots, r$ evaluated at a balance $\mathbf{c}$ are finite, not identically zero and independent. Then $m/(d-1)$ is a Kowalevsky exponent of the balance $\mathbf{c}$ with multiplicity at least r.

In fact Yoshida in [20] published Theorems 4 and 5 with m instead of $m/(d-1)$. Later on this was corrected, see [7, 8, 21].

3. Proof of Theorem 2

We divide the proof of Theorem 2 into different subsections. In the first subsection we study the polynomial first integrals.

3.1. The polynomial first integrals.

Now we shall compute the polynomial first integrals of the Hamiltonian systems with the homogeneous polynomial potentials given in Table 1.

The Hamiltonian system associated to the Hamiltonian (5) is

$$\dot{q}_i = p_i, \quad \dot{p}_i = -\frac{\partial V}{\partial q_i}, \qquad i = 1, 2. \tag{9}$$

If $V = V(q_1, q_2)$ is a homogeneous polynomial of degree 3, then this Hamiltonian system is a quasi–homogeneous polynomial differential system (8) with $\mathbf{x} = (q_1, q_2, p_1, p_2)$ and weight degree $d = 2$ with respect to the weight exponent

$\mathbf{s} = (s_1, s_2, s_3, s_4) = (2, 2, 3, 3)$. Hence we can apply the theory for finding polynomial first integrals to our Hamiltonian systems (9) described in the subsection 2.

The potential V_0 of Table 1 is

$$V_0 = V_0(q_1, q_2) = a(iq_1 + q_2)^3.$$

Unfortunately if we compute for the Hamiltonian system (9) with the potential $V = V_0$ its balances we see that this differential system has no balances. So the theory of quasi–homogeneous polynomial differential systems for finding polynomial first integrals cannot be applied to this Hamiltonian system. But it is easy to check that

$$H_0 = p_1 - p_2 i,$$

is a polynomial first integral for this Hamiltonian system.

The potential V_1 of Table 1 is

$$V_1 = V_1(q_1, q_2) = a(-iq_1 + q_2)(iq_1 + q_2)^2.$$

The Hamiltonian system (9) with the potential $V = V_1$ has the balance

$$\mathbf{c} = \left(\frac{3i}{2a}, -\frac{3}{2a}, -\frac{3i}{2a}, \frac{3}{a}\right).$$

The Kowalevsky exponents of this balance are -1, -1, 6 and 6. By Theorem 4 if $H_2(q_1, q_2, p_1, p_2)$ is a weighted homogeneous polynomial first integral of weight degree m with respect to the weight exponent $(2, 2, 3, 3)$ of the quasi–homogeneous polynomial differential system (9) with weight degree $d = 2$ with respect to the weight exponent $(2, 2, 3, 3)$, whose gradient evaluated at the balance $\mathbf{c}$ is finite and not identically zero, then $m/(d-1) = m$ is a Kowalevsky exponent. So if a such polynomial first integral exists its weighted degree is 6. Looking for these kind of polynomial first integral we obtain that

$$H_1 = 9ip_1^2 + 6p_1p_2 + 3ip_2^2 - 16aq_1^3 + 24iaq_1^2q_2 + 8iaq_2^3,$$

is a polynomial first integral of the Hamiltonian system (9) with $V = V_1$.

The potential V_2 of Table 1 is

$$V_2 = V_2(q_1, q_2) = a(-iq_1 + q_2)^2(iq_1 + q_2).$$

The Hamiltonian system (9) with the potential $V = V_2$ has the balance

$$\mathbf{c} = \left(-\frac{3i}{2a}, -\frac{3}{2a}, \frac{3i}{2a}, \frac{3}{a}\right).$$

The Kowalevsky exponents of this balance are -1, -1, 6 and 6. Again by Theorem 4, if the corresponding Hamiltonian system has a polynomial first integral under the assumptions of that theorem its weighted degree must be 6. Looking for these kind of polynomial first integral we obtain that

$$H_2 = -9ip_1^2 + 6p_1p_2 - 3ip_2^2 - 16aq_1^3 - 24iaq_1^2q_2 - 8iaq_2^3,$$

is a polynomial first integral of the Hamiltonian system (9) with $V = V_2$.

The potential V_3 of Table 1 is

$$V_3 = V_3(q_1, q_2) = a(-iq_1 + q_2)^3.$$

The Hamiltonian system (9) with the potential $V = V_3$ has no balances. But it is easy to check that

$$H_3 = p_1 + ip_2,$$

is a polynomial first integral of the Hamiltonian system (9) with $V = V_3$.

The potential V_4 of Table 1 is

$$V_4 = V_4(q_1, q_2) = q_1^3.$$

The Hamiltonian system (9) with the potential $V = V_4$ has the balance $\mathbf{c} = (-2, 0, 4, 0)$. The Kowalevsky exponents of this balance are -1, 2, 3 and 6. Again by Theorem 4, if the corresponding Hamiltonian system has a polynomial first integral under the assumptions of that theorem its weighted degree must be 2, 3 or 6. Looking for these kind of polynomial first integral we obtain that

$$H_4 = p_2, \quad \text{or} \quad \tilde{H}_4 = p_1^2 + 2q_1^3,$$

are polynomial first integrals of the Hamiltonian system (9) with $V = V_4$.

The potential V_5 of Table 1 is

$$V_5 = V_5(q_1, q_2) = \frac{1}{3}(q_1^3 + bq_2^3).$$

The Hamiltonian system (9) with the potential $V = V_5$ has the following three balances if $b \neq 0$ and only one $b = 0$:

$$\mathbf{c}_1 = (-6, 0, 12, 0), \quad \mathbf{c}_2 = \left(0, -\frac{6}{b}, 0, \frac{12}{b}\right), \quad \mathbf{c}_3 = \left(-6, -\frac{6}{b}, 12, \frac{12}{b}\right).$$

The Kowalevsky exponents of the first two balances are -1, 2, 3 and 6, and of the third balance are -1, -1, 6 and 6. Again by Theorem 4, if the corresponding Hamiltonian system has a polynomial first integral under the assumptions of that theorem its weighted degree must be 2, 3 or 6. Looking for these kind of polynomial first integral we obtain that

$$H_5 = 3p_1^2 + 2q_1^3, \quad \text{or} \quad \tilde{H}_5 = p_2^2 + 2q_2^3,$$

are polynomial first integrals of the Hamiltonian system (9) with $V = V_5$.

The potential V_6 of Table 1 is

$$V_6 = V_6(q_1, q_2) = \frac{1}{2}q_1^2 q_2 + q_2^3.$$

The Hamiltonian system (9) with the potential $V = V_6$ has the following three balances

$$\mathbf{c}_1 = (0, -2, 0, 4), \quad \mathbf{c}_2 = (12i, -6, -24i, 12), \quad \mathbf{c}_3 = (-12i, -6, 24i, 12).$$

The Kowalevsky exponents of the first balance is -1, 1, 4 and 6, and of the last two balances are -3, -1, 6 and 8. Again by Theorem 4, if the corresponding Hamiltonian system has a polynomial first integral under the assumptions of that theorem its weighted degree must be 4, 6 or 8. Looking for these kind of polynomial first integral we obtain that

$$H_6 = -8p_1p_2q_1 + 8p_1^2q_2 - q_1^2(q_1^2 + 4q_2^2),$$

is a polynomial first integral of the Hamiltonian system (9) with $V = V_6$.

The potential V_7 of Table 1 is

$$V_7 = V_7(q_1, q_2) = \frac{1}{2}q_1^2 q_2 + \frac{8}{3}q_2^3.$$

The Hamiltonian system (9) with the potential $V = V_7$ has the following three balances

$$\mathbf{c}_1 = \left(0, -\frac{3}{4}, 0, \frac{3}{2}\right), \ \mathbf{c}_2 = (6\sqrt{14}i, -6, -12\sqrt{14}i, 12), \ \mathbf{c}_3 = (-6\sqrt{14}i, -6, 12\sqrt{14}i, 12).$$

The Kowalevsky exponents of the first balance is -1, $3/2$, $7/6$ and 6, and of the last two balances are -7, -1, 6 and 12. Again by Theorem 4, if the corresponding Hamiltonian system has a polynomial first integral under the assumptions of that theorem its weighted degree must be 6 or 12. Looking for these kind of polynomial first integral we obtain that

$$H_7 = 72p_1^4 - 36p_1p_2q_1^3 - 3q_1^6 + 2(3p_2^2 + 16q_2^3)(3p_2^2 + 6q_1^2q_2 + 16q_2^3) + 12p_1^2(3p_2^2 + 12q_1^2q_2 + 16q_2^3),$$

is a polynomial first integral of the Hamiltonian system (9) with $V = V_7$.

The potential $V_8^{\pm}$ of Table 1 is

$$V_8^{\pm} = V_8^{\pm}(q_1, q_2) = \pm i\frac{\sqrt{3}}{18}q_1^3 + \frac{1}{2}q_1^2q_2 + q_2^3.$$

The Hamiltonian system (9) with the potential $V = V_8^{\pm}$ has the following three balances

$$\mathbf{c}_1 = (0, -2, 0, 4), \ \mathbf{c}_2 = (\pm 4\sqrt{3}i, -4, \mp 8\sqrt{3}i, 8), \ \mathbf{c}_3 = (\mp 24\sqrt{3}i, -18, \pm 48\sqrt{3}i, 36).$$

The Kowalevsky exponents of the first balance are -1, 1, 4 and 6, of the second are -2, -1, 6 and 7, and of the third are -7, -1, 6 and 12. Again by Theorem 4, if the corresponding Hamiltonian system has a polynomial first integral under the assumptions of that theorem its weighted degree must be 1, 4, 6 or 12. Looking for these kind of polynomial first integral we obtain that

$$\begin{aligned} H_8^{\pm} = \ & -q_1^6 \pm 6\sqrt{3}iq_1^5q_2 + 27q_1^4q_2^2 \pm 6\sqrt{3}iq_1^3(p_1^2 + p_2^2 + 2q_2^3) \\ & + 54q_1^2q_2(p_1^2 + p_2^2 + 2q_2^3) + 27(p_1^2 + p_2^2 + 2q_2^3)^2, \end{aligned}$$

is a polynomial first integral of the Hamiltonian system (9) with $V = V_8^{\pm}$.

The potential V_9 of Table 1 is

$$V_9 = \frac{a}{3}q_1^3 + \frac{1}{2}q_1^2q_2 + \frac{1}{6}q_2^3, \quad a \in \mathbb{R}.$$

The Hamiltonian system (9) with the potential $V = V_8$ has the following three balances

$$\begin{aligned} \mathbf{c}_1 &= (0, -12, 0, 24), \\ \mathbf{c}_2 &= \left(\frac{6}{\sqrt{a^2+1}}, -\frac{6a}{\sqrt{a^2+1}} - 6, -\frac{12}{\sqrt{a^2+1}}, \frac{12a}{1+\sqrt{a^2+1}}\right), \\ \mathbf{c}_3 &= \left(-\frac{6}{\sqrt{a^2+1}}, \frac{6a}{\sqrt{a^2+1}} - 6, \frac{12}{\sqrt{a^2+1}}, \frac{12a}{1-\sqrt{a^2+1}}\right). \end{aligned}$$

The Kowalevsky exponents of the first balance are -1, -1, 6 and 6, of the second and third balances are -1, 2, 3 and 6. Again by Theorem 4, if the corresponding Hamiltonian system has a polynomial first integral under the assumptions of that theorem its weighted degree must be 2, 3 or 6. Looking for these kind of polynomial first integral we obtain that

$$H_9 = \frac{2a^2q_1^3}{3} + \frac{q_1^3}{6} + aq_2q_1^2 + \frac{q_2^2q_1}{2} + ap_1^2 + p_1p_2,$$

is a polynomial first integral of the Hamiltonian system (9) with $V = V_9$.

3.2. **The Darboux polynomials with nonzero cofactor.** Now we shall compute the Darboux polynomials of system (1) with nonzero cofactor.

It was proved in [15] that there exists an affine change of variables in such a way that the Hamiltonian system (1) with homogeneous potential (4) can be written in the new variables as reduced as a Hamiltonian system (1) wtith homogeneous potential

$$V = \frac{a}{3}q_1^3 + \frac{1}{2}q_1^2 q_2 + \frac{b}{3}q_2^3, \quad a, b \in \mathbb{R},\ b \neq 0. \tag{10}$$

From now on, we will work with the homogenous potential (10).

Proposition 6. *The Hamiltonian system* (1) *with homogeneous potential* (10) *has no Darboux polynomials with nonzero cofactor.*

Proof. We consider a Darboux polynomial with non-zero cofactor. We write it as $f = \sum_{j=0}^{n} f_j(p_1, p_2, q_1, q_2)$ where each f_j is a homogeneous polynomial of degree j in the variables p_1, p_2, q_1, q_2. Without loss of generality we can assume that $f_n \neq 0$ with $n > 0$. We have

$$p_1\frac{\partial f}{\partial q_1} + p_2\frac{\partial f}{\partial q_2} - (aq_1^2 + q_1q_2)\frac{\partial f}{\partial p_1} - (q_1^2/2 + bq_2^2)\frac{\partial f}{\partial p_2} = Kf, \tag{11}$$

where $K = \alpha_0 + \alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 q_1 + \alpha_4 q_2$, with $\alpha_i \in \mathbb{C}$ not all zero.

We have that the terms of degree $n+1$ in (11) satisfy

$$-(aq_1^2 + q_1q_2)\frac{\partial f_n}{\partial p_1} - (q_1^2/2 + bq_2^2)\frac{\partial f_n}{\partial p_2} = (\alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 q_1 + \alpha_4 q_2) f_n. \tag{12}$$

Solving the differential equation in (12) we have

$$f_n = K_n\left(q_1, q_2, \frac{2q_1(aq_1 + q_2)p_2 - (q_1^2 + 2bq_2^2)p_1}{2q_1(aq_1 + q_2)}\right)\exp\left(\frac{-p_1 T_1}{4q_1^2(aq_1 + q_2)^2}\right) \tag{13}$$

where K_n is an arbitrary function and

$$\begin{aligned} T_1 = {} & 4\alpha_3 q_1^2 q_2 + 4\alpha_4 q_1 q_2^2 + 2\alpha_1 q_1 q_2 p_1 - \alpha_2(q_1^2 + 2bq_2^2)p_1 \\ & + 4\alpha_2 q_1 q_2 p_2 + 2aq_1^2(2\alpha_3 q_1 + 2\alpha_4 q_2 + \alpha_1 p_1 + 2\alpha_2 p_2). \end{aligned} \tag{14}$$

Let

$$Y = \frac{2q_1(aq_1 + q_2)p_2 - (q_1^2 + 2bq_2^2)p_1}{2q_1(aq_1 + q_2)} \quad \text{then} \quad p_2 = Y + \frac{p_1(q_1^2 + 2bq_2^2)}{2q_1(aq_1 + q_2)}.$$

Then we can rewrite (13) and (14) as

$$f_n = K_n(q_1, q_2, Y)\exp\left(\frac{-p_1\bar{T}_1}{4q_1^2(aq_1 + q_2)^2}\right)$$

with

$$\begin{aligned} \bar{T}_1 = {} & 2q_1q_2(\alpha_1 p_1 + 2\alpha_3 q_1 + 2\alpha_4 q_2) + \alpha_2(p_1(q_1^2 + 2bq_2^2) + 4q_1q_2 Y) \\ & + 2aq_1^2(\alpha_1 p_1 + 2(\alpha_3 q_1 + \alpha_4 q_2 + \alpha_2 Y)). \end{aligned}$$

Since f_n must be a polynomial and the function K_n in the variables q_1, q_2, p_1 and Y does not depend on p_1 we must have that $\bar{T}_1 = 0$. Setting the coefficient of Y in $\bar{T}_1$ equal to zero, we get $4\alpha_2 q_1(aq_1 + q_2) = 0$ and thus $\alpha_2 = 0$. Then, setting the coefficient of $q_1q_2^2$ equal to zero we get $4\alpha_4 = 0$ that is $\alpha_4 = 0$. Moreover, the coefficient of $q_1q_2p_1$ is $2\alpha_1$ and thus $\alpha_1 = 0$. Finally, setting the coefficient of $q_1^2q_2$

equal to zero we get $4\alpha_3 = 0$ that is $\alpha_3 = 0$. In short, $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$. This implies that $K = \alpha_0$.

Now computing the terms of degree 0 in (11) we get $\alpha_0 f_0 = 0$. Since $\alpha_0 \neq 0$ (otherwise $K = 0$ in contradiction with the fact that f is a Darboux polynomial with non-zero cofactor), we get $f_0 = 0$. Then the terms of degree 1 in (11) become

$$p_1 \frac{\partial f_1}{\partial q_1} + p_2 \frac{\partial f_1}{\partial q_2} = \alpha_0 f_1.$$

Solving it we get

$$f_1 = K_1\left(p_1, p_2, \frac{p_1 q_2 - p_2 q_1}{p_1}\right) \exp\left(\frac{\alpha_0 q_1}{p_1}\right). \tag{15}$$

Let

$$Q = \frac{p_1 q_2 - p_2 q_1}{p_1} \quad \text{and then} \quad q_2 = -Q + \frac{p_2 q_1}{p_1}.$$

In this new variable, we can rewrite (15) as

$$f_1 = K_1(p_1, p_2, Q) \exp\left(\frac{\alpha_0 q_1}{p_1}\right).$$

Since f_1 must be a polynomial and $\alpha_0 \neq 0$ we must have that $K_1 = 0$ and thus $f_1 = 0$. Proceeding inductively we get that $f_j = 0$ for $j = 2, 3, \ldots, n$ in contradiction with the fact that $f_n \neq 0$. This concludes the proof of the theorem. $\square$

Now we shall study the exponential factors of system (1). In view of subsections 3.1 and 3.2, and Theorem 1, the unique homogeneous potentials of the form (10) which admit an additional Darboux polynomial are the ones given in Table 1. Therefore, we will only study the systems with potentials that do not belong to Table 1. For these systems we know that the unique Darboux polynomial is the Hamiltonian H with cofactor zero.

3.3. **The exponential factors.** In this subsection we will prove the following result.

Proposition 7. *The unique exponential factors for the Hamiltonian system* (1) *with homogeneous potential* (10) *that do not belong to Table 1 are* $e^{P(H)/Q(H)}$ *(where* P *and* Q *are polynomials in the variable* H*),* e^{q_1}*,* e^{q_2} *and linear combinations of all the exponents in the exponential.*

To prove Proposition 7 we will use the following two known results whose proof and geometrical meaning is given in [3, 14].

Proposition 8. *The following statements hold.*

(a) *If* $E = \exp(g_0/g_1)$ *is an exponential factor for the polynomial system* (1) *and* g_1 *is not a constant polynomial, then* $g_1 = 0$ *is an invariant algebraic hypersurface.*
(b) *Eventually* e^{g_0} *can be an exponential factor, coming from the multiplicity of the infinite invariant hyperplane.*

The following result given in [14] characterizes the algebraic multiplicity of an invariant algebraic hypersurface using the number of exponential factors of system (1) associated with the invariant algebraic hypersurface.

Theorem 9. *Given an irreducible invariant algebraic hypersurface $g_1 = 0$ of degree m of system* (1)*, it has algebraic multiplicity k if and only if the vector field associated to system* (1) *has $k-1$ exponential factors of the form* $\exp(g_{0,i}/g_1^i)$*, where $g_{0,i}$ is a polynomial of degree at most im and $(g_{0,i}, g_1) = 1$ for $i = 1, \dots, k-1$.*

In view of Theorem 9 if we prove that e^{g_0/g_1} is not an exponential factor with degree $g_0 \le$ degree g_1, there are no exponential factors associated to the invariant algebraic curve $g_1 = 0$.

Proof of Proposition 7. Note that H is an irreducible polynomial in $\mathbb{C}[q_1, q_2, p_1, p_2]$. Indeed, $H = H_1 H_2$, and consequently H_1 and H_2 will be Darboux polynomials. By Proposition 6, H_1 and H_2 will be two polynomial first integrals of system (1) of degree at most 2, and such first integrals do not exist.

System (1) has the irreducible Darboux polynomial H. Then in view of Proposition 8 it can have an exponential factor of the form: either $E = \exp(g)$ with $g \in \mathbb{C}[p_1, p_2, q_1, q_2] \backslash \mathbb{C}$, or $E = \exp(g/H^m)$ with $m \ge 1$ and such that $g \in \mathbb{C}[p_1, p_2, q_1, q_2]$ is coprime with H.

We first prove that system (1) has no exponential factors of the form $E = \exp(g)$ with $g \in \mathbb{C}[p_1, p_2, q_1, q_2] \setminus \mathbb{C}$. In this case, g satisfies

$$p_1 \frac{\partial g}{\partial q_1} + p_2 \frac{\partial g}{\partial q_2} - (a q_1^2 + q_1 q_2) \frac{\partial g}{\partial p_1} - (q_1^2/2 + b q_2^2) \frac{\partial g}{\partial p_2} = L,$$

where we have simplified the common factor $E = \exp(g)$ and where $L = \beta_0 + \beta_1 p_1 + \beta_2 p_2 + \beta_3 q_1 + \beta_4 q_2$ with $\beta_i \in \mathbb{C}$. We write $g = h + \beta_1 q_1 + \beta_2 q_2$ for some $h \in \mathbb{C}[p_1, p_2, q_1, q_2]$. Then h satisfies

$$p_1 \frac{\partial h}{\partial q_1} + p_2 \frac{\partial h}{\partial q_2} - (a q_1^2 + q_1 q_2) \frac{\partial h}{\partial p_1} - (q_1^2/2 + b q_2^2) \frac{\partial h}{\partial p_2} = \beta_0 + \beta_3 q_1 + \beta_4 q_2. \tag{16}$$

Evaluating (16) on $p_1 = p_2 = q_1 = q_2 = 0$ we get $\beta_0 = 0$. We write h as $h = \sum_{j=0}^n h_j(p_1, p_2, q_1, q_2)$ where each h_j is a homogeneous polynomial of degree j in each variables p_1, p_2, q_1 and q_2. Now computing the terms of degree one in (16) we get

$$p_1 \frac{\partial h_1}{\partial q_1} + p_2 \frac{\partial h_1}{\partial q_2} = \beta_3 q_1 + \beta_4 q_2.$$

Clearly, setting $p_1 = p_2 = 0$ in the above equality we get $\beta_3 = \beta_4 = 0$. In short, h satisfies

$$p_1 \frac{\partial h}{\partial q_1} + p_2 \frac{\partial h}{\partial q_2} - (a q_1^2 + q_1 q_2) \frac{\partial h}{\partial p_1} - (q_1^2/2 + b q_2^2) \frac{\partial h}{\partial p_2} = 0.$$

In other words, h is either a constant or a polynomial first integral of system (1). By hypothesis we have that $h = h(H)$ where h can be constant. Hence, in this case

$$g = h(H) + \beta_1 q_1 + \beta_2 q_2, \quad \beta_1, \beta_2 \in \mathbb{C}.$$

Assume now that system (1) has an exponential factor of the form $E = \exp(g/H^m)$ with $m \ge 1$ such that H and $g \in \mathbb{C}[p_1, p_2, q_1, q_2]$ are coprime. In view of Theorem 9 we can assume that $m = 1$ and that g has degree at most three (note that here $g_1 = H$ has degree three). We write g as a polynomial of degree three in the variables p_1, p_2, q_1, q_2 and since $e^{g/H}$ must be an exponential factor g must satisfy

$$
\begin{aligned}
&p_1\frac{\partial g}{\partial q_1} + p_2\frac{\partial g}{\partial q_2} - (aq_1^2 + q_1q_2)\frac{\partial g}{\partial p_1} - (q_1^2/2 + bq_2^2)\frac{\partial g}{\partial p_2} \\
&= (\beta_0 + \beta_1 p_1 + \beta_2 p_2 + \beta_3 q_1 + \beta_4 q_4)\left(\frac{p_1^2 + p_2^2}{2} + \frac{a}{3}q_1^3 + \frac{1}{2}q_1^2 q_2 + \frac{b}{3}q_2^3\right).
\end{aligned} \tag{17}
$$

Computing the solutions of (17) using an algebraic manipulator such as mathematica and recalling that by assumptions we have $b \neq 0$ we get that $\beta_i = 0$ for $i = 0, \ldots, 4$, i.e. $L = 0$ and g is either const H, or $g = \mathrm{const}/H$, or $g = H_9$. So, Proposition 7 is proved. □

Now we have all the ingredients to prove Theorem 2

3.4. **Proof of Theorem 2.** In order to proof Theorem 2 we need the following result whose proof is given in [6].

Theorem 10. *Suppose that system* (1) *admits p Darboux polynomials and with cofactors K_i and q exponential factors F_j with cofactors L_j. Then there exists $\lambda_j, \mu_j \in \mathbb{C}$ not all zero such that*

$$\sum_{i=1}^{q} \lambda_k K_i + \sum_{i=1}^{q} \mu_i L_i = 0$$

if and only if the function G given in (3) *(called of Darboux type) is a first integral of system* (1).

In view of Theorem 10 to characterize the Darboux first integrals we need to compute the Darboux polynomials and the exponential factors. Then, under our assumptions, using Propositions 6 and 7 if G is a Darboux first integral of system (1) it must be of the form (3), i.e., $G = P_1(H)e^{\lambda P(H)/Q(H) + \mu_1 q_1 + \mu_2 q_2}$ for some P_1, P, Q polynomials and $\lambda, \mu_1, \mu_2 \in \mathbb{C}$. Moreover, the cofactors must satisfy

$$\mu_1 p_1 + \mu_2 p_2 = 0.$$

Solving this system we have $\mu_1 = \mu_2 = 0$ which shows that $G = P_1(H)e^{\lambda P(H)/Q(H)}$ only depends on H. This concludes the proof of Theorem 2.

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